International Journal of Theoretical Physics, Vol. 9, No. 4 (1974), pp. 269–276

Equation of Motion for Non-Geodesic Laboratory Bodies†

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Received: 4 *July* 1973

Abstract

The gravitational equation of motion of laboratory bodies made up of electrically interacting molecules, the bodies being coupled to non-geodesic laboratories, is obtained for metrical theories of gravity. Application is made to the experiment of Witteborn and Fairbank in which electrons or positrons are 'dropped' inside a conducting shield. We show that the inertial and gravitational weight of a body depends on the location of the supporting force, and that a laboratory body, in general, possesses an inertial or gravitational mass *tensor* which differs from the body's energy content divided by the speed of light squared.

1. *Introduction*

Previously the passive gravitational mass (M_G) and inertial mass (M_I) have separately been calculated for realistic laboratory bodies (Nordtvedt, 1970). In that calculation we assumed the general class of metric theories of gravity (Nordtvedt, 1968). One of our results, that the ratio $M_G/M_I = 1$, could in retrospect be obtained quite simply. If we transform to a geodesic coordinate system which is free-falling at the rate g, the metric field $g_{\mu\nu}(\mathbf{r}, t)$ takes locally the Minkowski form. The equations of motion of the interacting matter which makes up the laboratory body are then those of special relativity, in which a free body will experience non-accelerative motion.

In this paper we will consider situations where the problem is not simplified by going to a geodesic (free-falling) coordinate system—situations where the body is not free but rather is coupled in some way to a non-geodesic laboratory. We can imagine two classes of such experiments: (1) bodies can be held at rest in the laboratory ('weighing experiments') and (2) bodies can be allowed to fall and compared with the free-fall rate g.

Our calculational results will be applicable to the analysis of 'free-fall' experiments with charged elementary particles (Witteborn & Fairbank, 1967), to

† Work supported by National Aeronautics and Space Administration under Grant No. NGR 27-001-040.

‡ Alfred P. Sloan Research Fellow.

experiments which 'weigh' bodies in gravitational or acceleration fields, and to envisioning possible variations on the Eötvös-type experiments.

Here again we assume the validity of the general class of metric theories of gravity; also gravitational gradients are neglected. The results will consequently exhibit a complete equivalence between gravitational acceleration and acceleration of the coordinate frame (Einstein's equivalence principle).

2. *The Metric for Local Physics*

Assuming the metric class of gravitational theories and a quasi-static distribution of massive objects m_i at locations $\mathbf{r}_i$, a metric field $g_{\mu\nu}(\mathbf{r}, t)$ leads to an invariant line element

$$ds^2 \equiv g_{\mu\nu}(\mathbf{r}, t)\, dx^\mu\, dx^\nu, \tag{2.1}$$

$\mu = 0, 1, 2, 3$ (ct, x, y, z), which takes the form in isotropic coordinates

$$ds^2 = A(\mathbf{r}, m_i, \mathbf{r}_i)\, dt^2 - B(\mathbf{r}, m_i, \mathbf{r}_i)\, dr^2 \tag{2.2}$$

where A and B are functions of the field point $\mathbf{r}$ and the mass distribution $(m_i, \mathbf{r}_i)$.

Expanding A and B in Taylor series

$$A(\mathbf{r}) = A_0 + \mathbf{A}_1 . \mathbf{r} + O(r^2) \ldots, \tag{2.3a}$$

$$B(r) = B_0 + \mathbf{B}_1 . \mathbf{r} + O(r^2) \ldots, \tag{2.3b}$$

we can then use the coordinate freedom of metric theories to rescale the time and space coordinates:

$$\sqrt{(A_0)}t \equiv t', \tag{2.4a}$$

$$\sqrt{(B_0)}\mathbf{r} \equiv \mathbf{r}', \tag{2.4b}$$

which puts the line element into the form (we hereafter use units in which $c = 1$)

$$ds^2 = (1 - 2g . r)\, dt^2 - (1 + 2\gamma g . r)\, dr^2 \tag{2.5}$$

with $\mathbf{g}$ identified as the local gravitational acceleration and γ as a theory-dependent (and gauge-dependent) dimensionless number. An additional purely spatial coordinate transformation

$$\mathbf{r}' \equiv \mathbf{r} - \tfrac{1}{2}\gamma r^2 \mathbf{g} + \gamma \mathbf{g} . \mathbf{r}\mathbf{r} \tag{2.6}$$

finally brings the line element to the simple form

$$ds^2 = (1 - 2\mathbf{g} . \mathbf{r})\, dt^2 - dr^2. \tag{2.7}$$

This metric (2.7) can also be obtained by accelerating an inertial coordinate frame; under the transformation

$$t' = t(1 + g . r), \tag{2.8a}$$

$$\mathbf{r}' = \mathbf{r} + \tfrac{1}{2}\mathbf{g}t^2. \tag{2.8b}$$

The Minkowski line element

$$ds^2 = dt^2 - dr^2 \tag{2.9}$$

takes the form (2.7). This duality of interpretation of the physical origin of the line element (2.7) is the mathematical basis of Einstein's equivalence principle and is incorporated into the foundations of metric theories of gravity.

Our ability to eliminate the spatial metric field potential via the coordinate transformation (2.6) leads to the theorem:

No local gravitational experiment in which the gravitational gradients are neglected can measure the theory-dependent parameter γ.

3. *Equation of Motion of Charged Particles*

Using previous results (Nordtvedt, 1970) the Maxwell fields in a metric gravitational field given by (2.7) are

$$\phi(\mathbf{r}) = \sum_i \frac{e_i}{|\mathbf{r} - \mathbf{r}_i|}\,[1 - \tfrac{1}{2}\mathbf{g}.(\mathbf{r} + \mathbf{r}_i)] - \tfrac{1}{2}\sum_i e_i \frac{(\mathbf{r} - \mathbf{r}_i)}{|\mathbf{r} - \mathbf{r}_i|}\,.\,\mathbf{a}_i + \ldots, \tag{3.1a}$$

and

$$\bar{A}(\mathbf{r}) = \sum_i \frac{e_i}{|r - r_i|}\,\mathbf{v}_i + \ldots, \tag{3.1b}$$

in which charges e_i are traveling at velocities $\mathbf{v}_i$ and acceleration $\mathbf{a}_i$. Also we have found the equation of motion of a charged particle to sufficient accuracy,

$$m\mathbf{a} + \tfrac{1}{2}mv^2\mathbf{a} + m\mathbf{v}\,.\,\mathbf{a}\mathbf{v} + \frac{d}{dt}(mg\,.\,rv) = m\mathbf{g} + \tfrac{1}{2}mv^2g - e(\nabla\phi + \dot{\mathbf{A}}), \tag{3.2}$$

neglecting magnetic forces. The electrical force term in (3.2) can be evaluated using (3.1a, b):

$$-(\nabla\phi + \dot{\mathbf{A}}) = \sum_i e_i \frac{\mathbf{r} - \mathbf{r}_i}{|\mathbf{r} - \mathbf{r}_i|^3}\,(1 - \mathbf{g}\,.\,\mathbf{r}) + \tfrac{1}{2}\sum_i \frac{e_i}{|\mathbf{r} - \mathbf{r}_i|}\,(\mathbf{g} - \mathbf{a}_i)$$

$$+ \tfrac{1}{2}\sum_i \frac{e_i}{|\mathbf{r} - \mathbf{r}_i|^3}\,(\mathbf{r} - \mathbf{r}_i)\,.\,(\mathbf{g} - \mathbf{a}_i)(\mathbf{r} - \mathbf{r}_i) + \ldots \tag{3.3}$$

Applying (3.2) and (3.3) to the case of a charged elementary particle (m, e) instantaneously at rest in a laboratory which is itself accelerating at rate $\mathbf{a}_L$ gives

$$\mathbf{a} = \mathbf{g} + \frac{e}{m}\mathbf{E} + \frac{e}{2m}\sum_i \frac{Q_i}{|\mathbf{r} - \mathbf{r}_i|}\,(\mathbf{g} - \mathbf{a}_L)$$

$$+ \frac{e}{2m}\sum_i \frac{Q_i}{|r - r_i|^3}\,(\mathbf{r} - \mathbf{r}_i)(\mathbf{r} - \mathbf{r}_i)\,.\,(\mathbf{g} - \mathbf{a}_L); \tag{3.4}$$

$\mathbf{E}$ is the laboratory electrical field, Q_i are the laboratory charges. (3.4) loses its anomalous terms when the laboratory is in free-fall ($\mathbf{a}_L = \mathbf{g}$).

4. *Equation of Motion of Laboratory Bodies*

Consider now a neutral, macroscopic laboratory body viewed as an assembly of interacting charged particles (an atomic and solid-state model of matter). We allow the body to be in interaction with the laboratory via electrical forces of both a short-range type (solid state contact forces) and of a long-range type (e.g. polarization forces from non-contacting capacitor plates). The electrical fields (3.3) which each charged particle in the body experiences will result from both external laboratory charges Q_i and other internal charges in the body e_i. To obtain an equation of motion for the body then, an appropriate sum of (3.2) over all particles in the body must be made.

We will need to apply a virial relation for the body experiencing external forces,

$$\sum_i m_i \mathbf{v}_i \cdot \mathbf{a}\mathbf{v}_i + \tfrac{1}{2}\sum \frac{e_i e_i}{\mathbf{r}_{ij}{}^3}\, r_{ij} \cdot \mathbf{a}\mathbf{r}_{ij} = -\sum_i \mathbf{f}_i(\mathrm{ex})\,\mathbf{a}\cdot\mathbf{r}_i, \tag{4.1}$$

when $\mathbf{f}_i(\mathrm{ex})$ is the external electrical force acting on the ith particle of the body. We assume no net torque acting on the body.

Then by use of (3.2), (3.3), and (4.1) a body's acceleration becomes

$$M\mathbf{a} - \sum_i \mathbf{f}_i(\mathrm{cx})\mathbf{a}\cdot\mathbf{r}_i = M\mathbf{g} - \sum_i \mathbf{f}_i(\mathrm{ex})\mathbf{g}\cdot\mathbf{r}_i + \sum_i e_i \mathbf{E}(\mathbf{r}_i)$$
$$+ \tfrac{1}{2}\sum_{ij}\frac{Q_i e_i}{r_{ij}}(\mathbf{g}-\mathbf{a}_L) + \tfrac{1}{2}\sum_{ij}\frac{Q_i e_j}{r_{ij}{}^3}\,\mathbf{r}_{ij}\mathbf{r}_{ij}\cdot(\mathbf{g}-\mathbf{a}_L), \tag{4.2}$$

where M is the body's energy content;

$$M \equiv \sum_i (m_i + \tfrac{1}{2} m_i v_i{}^2) + \tfrac{1}{2}\sum_{ij}\frac{e_i e_j}{r_{ij}}, \tag{4.3}$$

and $\mathbf{E}(\mathbf{r})$ is the laboratory electrical field,

$$\mathbf{E}(\mathbf{r}) \equiv \sum_i \frac{Q_i}{|\mathbf{r}-\mathbf{r}_i|^3}(\mathbf{r}-\mathbf{r}_i). \tag{4.4}$$

Note that even in the absence of gravity or an accelerated laboratory, the body possesses an inertial mass tensor:

$$M'_{\alpha\beta} = M\delta_{\alpha\beta} - \sum_i [f_i(\mathrm{ex})]_\alpha [r_i]_\beta \tag{4.5}$$

and hence the simple prescription of special relativity for inertial mass—$E = Mc^2$—loses its validity.

If we consider a 'weighing' experiment ($a_L = 0$ and the demand $a = 0$), (4.2) gives us the required supporting force:

$$\sum_i e_i \mathbf{E}(\mathbf{r}_i)[1 - \mathbf{g}\cdot\mathbf{r}_i] = -M\mathbf{g} - \tfrac{1}{2}\sum_{ij}\frac{Q_i e_j}{r_{ij}}\mathbf{g} - \tfrac{1}{2}\sum_{ij}\frac{Q_i e_j}{r_{ij}^3}\mathbf{r}_{ij}\cdot\mathbf{g}\mathbf{r}_{ij}. \quad (4.6)$$

Short-range (solid state) external electrical forces and long-range forces in (4.6) participate differently; this can be illustrated by considering a neutral, polarizable body:

$$\sum_j e_j = 0, \quad (4.7a)$$

$$\sum_j e_j \mathbf{r}_j \equiv \mathbf{p}, \quad (4.7b)$$

and

$$\mathbf{E}(r) = \mathbf{E}_{SR}(\mathbf{r}) + \mathbf{E}_{LR}^{(r)}, \quad (4.7c)$$

then (4.6) takes the form:

$$\int \rho(\mathbf{x})\mathbf{E}_{SR}^{(\mathbf{x})}(1 - \mathbf{g}\cdot\mathbf{x})\,d^3x + \mathbf{p}\cdot\nabla\mathbf{E}_{LR}$$
$$= -M\mathbf{g} + \tfrac{1}{2}(\mathbf{p}\cdot\mathbf{E}_{LR}\mathbf{g} + \mathbf{p}\cdot\mathbf{g}E_{LR} - \mathbf{E}_{LR}\cdot\mathbf{g}\mathbf{p}) - \tfrac{3}{2}\sum\frac{Q_i}{r_i^5}\mathbf{r}_i\cdot\mathbf{g}\mathbf{r}_i\cdot\mathbf{p}\mathbf{r}_i, \quad (4.8)$$

or on defining the potential

$$W(\mathbf{r}) \equiv \sum_i \frac{Q_i}{|r - r_i|^3}(\mathbf{r} - \mathbf{r})\cdot\mathbf{g}(\mathbf{r} - \mathbf{r}_i)\cdot\mathbf{p},$$

(4.8) can be written

$$\int \rho(\mathbf{x})\mathbf{E}_{SR}(\mathbf{x})[1 - g\cdot x]\,d^3x + \mathbf{p}\cdot\nabla\mathbf{E}_{LR}$$
$$= -(M - p\cdot E_{LR})\mathbf{g} + \tfrac{1}{2}\mathbf{p}\cdot\mathbf{g}\mathbf{E}_{LR} + \tfrac{1}{2}\nabla W. \quad (4.9)$$

5. *Electron-Positron Free-Fall Experiment*

Recurring suggestions are made that anti-particles might accelerate oppositely (or differently) from particles in a gravitational field, though most all theory contradicts this, and it has been argued (Schiff) that this would produce gravitational to inertial mass ratio variations in different atoms because of the presence of 'virtual' electron-positron pairs in the atomic and nuclear electric fields.

Experiments have been designed to measure the gravitational acceleration of electrons and anti-electrons (positrons) (Witteborn & Fairbank, 1967). Free electrons are 'dropped' inside a conducting metallic cylinder (metallic to shield out electric fields), the electrons being found to fall at less than 0·09*g*, experimental error compatible with no acceleration. This confirms calculations

(Schiff & Barnhill, 1966) which show that a gravitationally stressed conductor will produce an interior electric field necessary to support its conduction electrons in the gravitational field, $\mathbf{E} = (m/e)\mathbf{g}$. The same experiment should then find positrons to fall at the rate $2\mathbf{g}$.

Consider electrically charging the conducting shield up to some high potential comparable with the electron rest energy, 0·5 MeV. (3.4) gives the predicted acceleration of the electron:

$$\mathbf{a} = \mathbf{g} + \frac{e}{m}\sum \frac{Q_i(\mathbf{r}-\mathbf{r}_i)}{|r-r_i|^3} + \frac{e}{2m}\sum \frac{Q_i}{|\mathbf{r}-\mathbf{r}_i|}\mathbf{g}$$
$$+ \frac{e}{2m}\sum \frac{Q_i}{|\mathbf{r}-\mathbf{r}_i|^3}(\mathbf{r}-\mathbf{r}_i).\mathbf{g}(\mathbf{r}-\mathbf{r}). \tag{5.1}$$

The acceleration field $\mathbf{a}$ of (5.1) fulfills the two exact conditions

$$\nabla . \mathbf{a} = 4\pi\, e\rho/m, \tag{5.2a}$$

and

$$\nabla \times \mathbf{a} = -\frac{e}{m}\sum_i \frac{Q_i}{|r-r_i|^3}(\mathbf{r}-\mathbf{r}_i)\times \mathbf{g}. \tag{5.2b}$$

$\mathbf{a}$ must vanish throughout a conducting shield in order that the conduction electron density be in static equilibrium, and the free-fall electron experiment measures $\mathbf{a}$ in the interior cavity of the conducting shield.

Expanding the charge density in powers of $|\mathbf{g}|$, (5.1) becomes in first order:

$$\frac{e}{m}\sum_i \frac{Q_i^{(1)}}{|\mathbf{r}-\mathbf{r}_i|^3}(\mathbf{r}-\mathbf{r}_i) = -\mathbf{g}\left[1 + \frac{e}{2m}\sum_i \frac{Q_i^{(0)}}{|\mathbf{r}-\mathbf{r}_i|}\right]$$
$$-\frac{e}{2m}\sum_i \frac{Q_i^{(0)}}{|\mathbf{r}-\mathbf{r}_i|^3}(\mathbf{r}-\mathbf{r}_i)(\mathbf{r}-\mathbf{r}_i).\mathbf{g}. \tag{5.3}$$

For a conducting sphere of radius R and charged to potential

$$V \equiv Q^{(0)}/R,$$

(5.3) is fulfilled by an induced surface charge density

$$\sigma(\theta) = \frac{3}{4\pi}\frac{m}{e}\mathbf{g}.\mathbf{R}^2\left[1 + \frac{5}{6}\frac{eQ^{(0)}}{mR}\right]. \tag{5.4}$$

From (5.2a, b) it can be shown that there will be no acceleration of an electron to first order in $\mathbf{g}$ for locations in the interior of a conducting shield, regardless of the charge placed on the conductor. (5.2b) indicates that the tangential component of $\mathbf{a}$ is continuous from the conducting shield into the interior cavity; therefore the cavity boundary has no tangential acceleration field. Writing $\mathbf{a}$ as

$$\mathbf{a} = \nabla\chi + \nabla \times \mathbf{b}$$

gives

$$\nabla^2 \chi = \frac{4\pi e}{m} \rho \tag{5.5a}$$

and

$$\nabla^2 \mathbf{b} = \frac{e}{m} \sum_i \frac{Q_i}{|\mathbf{r} - \mathbf{r}_i|^3} (\mathbf{r} - \mathbf{r}_i) \times \mathbf{g}. \tag{5.5b}$$

The source of $\mathbf{b}$ is of second or higher order in $\mathbf{g}$ as

$$\sum_i \frac{Q_i}{|\mathbf{r} - \mathbf{r}_i|^3} (\mathbf{r} - \mathbf{r}_i)$$

vanishes inside a conductor to zeroth order in $\mathbf{g}$. Since therefore the tangential variations in χ are of second or higher order in $\mathbf{g}$, and in the cavity $\nabla^2 \chi = 0$, the radial variations of χ must also vanish to order $|\mathbf{g}|$.

Consequently no observable (linear in $\mathbf{g}$) effect can be produced by highly charging up ($eV \wedge mc^2$) the apparatus of Fairbank and Witterborn, in spite of the fact that we greatly rearrange [see (5.4)] the gravitationally induced electron density thereby.

6. *Summary*

We have studied the equation of motion of realistic laboratory bodies which are *not* in geodesic motion, with the goal of finding observable relativistic effects which are consequences of Einstein's equivalence principle.

It was found that the inertial mass differs from the body's energy content in cases where external forces are applied to the body. The inertial mass becomes the mass tensor (4.5).

Similarly, the gravitational mass differs from the energy content of the body, and hence the weight of a body (the force necessary to support the body in a gravitational field) is dependent on the location of the supporting forces as given in (4.6). Elsewhere (Nordtvedt, 1973) it has been shown that this result is simply derivable from considerations of energy conservation and the equivalence principle result that photons are energy-shifted when changing altitude in a gravitational or acceleration field.

Considering the effects of the coupled gravitational electric fields on conduction electrons, we have found that the induced electron density in a gravitationally supported or accelerated conducting shield is dependent on the electric potential of the shield; however free-fall rates of electrons in an interior cavity are unaffected to linear order in $\mathbf{g}$.

All results of this paper are valid within the complete class of metric theories of gravity and in the equivalence principle approximation. Therefore the effects are identically present in both gravitational fields and in accelerated coordinate frames.

References

Nordtvedt, K. Jr. (1968). *Physical Review*, **169** (5), 1017.
Nordtvedt, K. Jr. (1970). *International Journal of Theoretical Physics*, Vol. 3, No. 2, p. 133.
Nordtvedt, K. Jr. (1973). Submitted to *American Journal of Physics*.
Schiff, L. I. and Barnhill, M. V. (1966). *Physical Review*, **151**, 1067.
Schiff, L. I., private communication.
Witteborn, F. C. and Fairbank, W. M. (1967). *Physical Review Letters*, **19**, 1049.